

DEFORMATION OF MATRIX-VALUED ORTHOGONAL POLYNOMIALS RELATED TO GELFAND PAIRS

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ABSTRACT. In this paper we present a method to obtain deformations of families of matrix-valued orthogonal polynomials that are associated to the representation theory of compact Gelfand pairs. These polynomials have the Sturm-Liouville property in the sense that they are simultaneous eigenfunctions of a symmetric second order differential operator and we deform this operator accordingly so that the deformed families also have the Sturm-Liouville property. Our strategy is to deform the system of spherical functions that is related to the matrix-valued orthogonal polynomials and then check that the polynomial structure is respected by the deformation. Crucial in these considerations is the full spherical function Ψ_0 , which relates the spherical functions to the polynomials. We prove an explicit formula for Ψ_0 in terms of Krawtchouk polynomials for the Gelfand pair $(\mathrm{SU}(2) \times \mathrm{SU}(2), \mathrm{diag}(\mathrm{SU}(2)))$. For the matrix-valued orthogonal polynomials associated to this pair, a deformation was already available by different methods and we show that our method gives same results using explicit knowledge of Ψ_0 .

Furthermore we apply our method to some of the examples of size 2×2 for more general Gelfand pairs. We prove that the families related to the groups $\mathrm{SU}(n)$ are deformations of one another. On the other hand, the families associated to the symplectic groups $\mathrm{Sp}(n)$ give rise to a new family with an extra free parameter.

1. INTRODUCTION AND STATEMENT OF RESULTS

It is well known that the classical orthogonal polynomials are characterized by the property that their derivatives are also orthogonal polynomials, see e.g. [12]. One can exploit this characterization, for instance, to construct the Gegenbauer polynomials $C_n^{(\nu)}(y)$, for integer values of ν , by repeated differentiation of the Chebyshev polynomials $U_n(y) = C_n^{(1)}(y)$. In this way, the orthogonality relations, differential equations and other properties of the Gegenbauer polynomials can be obtained from those of the Chebyshev polynomials. In this paper we deal with the analogous construction for matrix-valued orthogonal polynomials.

For $N \in \mathbb{N}$, let $W : (a, b) \rightarrow \mathbb{C}^{N \times N}$ be a positive definite smooth weight matrix with finite moments. We consider the sesqui-linear pairing defined for a pair of matrix polynomials $P, Q \in \mathbb{C}^{N \times N}[y]$ by the matrix

$$\langle P, Q \rangle_W = \int_I P(y)W(y)Q(y)^* dy, \quad (1.1)$$

where $Q(x)^*$ denotes the conjugate transpose of $Q(x)$. We say that $(P_d)_d$, $P_d \in \mathbb{C}^{N \times N}[x]$, $d \in \mathbb{N}_0$, is a sequence of matrix-valued orthogonal polynomials (MVOPs from now on) with respect to (1.1) if

- (1) $\langle P_d, P_{d'} \rangle_W = 0$ if $d \neq d'$,
- (2) $\deg P_d = d$, for all $d \in \mathbb{N}_0$,
- (3) the leading coefficient of P_d is invertible for all $d \in \mathbb{N}$.

A sequence of monic MVOPs can be obtained by applying the Gram-Schmidt process on the ordered basis (I, yI, y^2I, \dots) , where I denotes the $N \times N$ identity matrix.

We say that the differential operator $D : \mathbb{C}^{N \times N}[y] \rightarrow \mathbb{C}^{N \times N}[y]$ is symmetric with respect to the matrix weight W if it satisfies $\langle PD, Q \rangle = \langle P, QD \rangle$ for all matrix polynomials P, Q . Any sequence $(P_d)_d$ of MVOPs with respect to W is a basis of the space of matrix polynomials, so that the

symmetry condition is equivalent to

$$\langle P_d D, P_{d'} \rangle = \langle P_d, P_{d'} D \rangle, \quad \text{for all } d, d' \in \mathbb{N}_0. \quad (1.2)$$

A pair (W, D) consisting of a matrix weight W together with a matrix-valued differential operator

$$D = \partial_y^2 F_2 + \partial_y F_1 + F_0,$$

where F_i is a polynomial of degree at most i , which is symmetric with respect to $\langle \cdot, \cdot \rangle$ is called a matrix-valued classical pair (MVCP from now on). Given a MVCP (W, D) , any sequence of MVOPs with respect to W is a family of simultaneous eigenfunctions of D , see [11, Proposition 2.10].

Definition 1.1. *A deformation of a MVCP (W, D) is a family $(W^{(\kappa)}, D^{(\kappa)})_{\kappa \in K}$ of MVCPs, where $K \subset \mathbb{R}$ is an open interval, so that $(W, D) = (W^{(\kappa_0)}, D^{(\kappa_0)})$ for some $\kappa_0 \in K$.*

We say that a deformation $(W^{(\kappa)}, D^{(\kappa)})_{\kappa \in K}$ allows for the shift ∂_y , if for a family of MVOPs $(P_d^{(\kappa)})_d$ of $W^{(\kappa)}$, we have that $(\partial_y P_d^{(\kappa)})_d$ is a family of MVOPs for $W^{(\kappa+1)}$.

The goal of this paper is to present a method to find deformations that allow for the shift ∂_y , of MVCPs (W, D) that are associated to compact Gelfand pairs of rank one [15, 21]. We also present a Rodrigues type formula for these cases. We apply our method to some examples of size 2×2 taken from [22], where we observe that some of the families actually fit in a single deformation, that moreover, allows for the shift ∂_y .

An earlier result in this direction is the case studied in [17], where sequences of matrix-valued Chebyshev polynomials [18, 19] are deformed into matrix-analogues of Gegenbauer polynomials. However, those results are based on a decomposition for the weight matrix that is quite particular for the Gelfand pair $(\text{SU}(2) \times \text{SU}(2), \text{diag}(\text{SU}(2)))$. If we apply our method to this case, we obtain the same deformations.

MVOPs associated to representation theory. The theory of MVOPs dates back to the work of M.G. Krein in the 1940s. Since then, the theory has been developed and connected to different fields such as scattering theory and spectral analysis, see for instance [1, 5, 6, 7]. In [3] A. Durán raises the question of whether it is possible to construct a sequence of MVOPs together with a matrix valued differential operator which has the MVOPs as simultaneous eigenfunctions. The first results in this direction are given in [9] from the study of matrix-valued spherical functions on $\text{SU}(3)/\text{U}(2)$. The link between matrix-valued spherical functions and MVOPs has been developed in several papers following [20, 9], see for instance [18, 19, 24]. This led to a uniform construction of MVOPs for compact Gelfand pairs (G, K) of rank one [15, 21, 22]. The families of MVOPs obtained from representation theory have many interesting properties and, in some cases, can be described in great detail. Certain families of MVOP related to the pair $(\text{SU}(n+1), \text{U}(n))$ have been exploited to derive stochastic models, see [8, 10, 13]. The family described in Section 3 leads to models of continuous-time bivariate Markov processes which are analyzed in detail in [14].

In this paper we deal with the families of MVOPs obtained in [15, 21, 22]. We develop a method to deform these families that relies in a specific decomposition of the weight matrix and the differential operator, see (1.7) and (1.4). The same decomposition arises in families of MVOPs constructed independently from group theory as well, see for instance [4, (4.8)]. Our method could also be carried out in these cases, to investigate the existence of deformations. The construction of MVOPs that we consider in this paper applies to a triple (G, K, μ) , where (G, K) is a compact Gelfand pair of rank one and μ is a suitable representation of K , see [22, Table 1]. The output is a family of $\mathbb{C}^{N_\mu \times N_\mu}$ -valued functions $\{\Psi_d^\mu : d \in \mathbb{N}_0\}$, defined on the interval $[0, 1]$, together with a matrix-valued differential operator Ω^μ for which the Ψ_d^μ are eigenfunctions. The function Ψ_d^μ is called the *full spherical function* of type μ and degree d . It follows from [15, 21] or [22, Theorem 2.7] that

$$\Psi_d^\mu(y) = P_d^\mu(y) \Psi_0^\mu(y), \quad y \in [0, 1],$$

for all $d \in \mathbb{N}$, where P_d^μ is a polynomial of degree d .

As a consequence of the Schur orthogonality relations, the full spherical functions $(\Psi_d^\mu)_d$ are pairwise orthogonal. More precisely, in [15, 21] it is proven that

$$\int_0^1 \Psi_d^\mu(y) T^\mu (\Psi_{d'}^\mu(y))^* (1-y)^\alpha y^\beta dy = 0, \quad d \neq d', \quad (1.3)$$

where $(1-y)^\alpha y^\beta$ is the ordinary Jacobi weight on the interval $[0, 1]$ that is associated to the Riemann symmetric space G/K and T^μ is a constant diagonal matrix. The spherical functions are eigenfunctions of the Casimir operator of the group G . This implies that the full spherical function Ψ_d^μ is an eigenfunction of a single variable differential operator Ω^μ , the radial part of the Casimir operator. In other words there is an operator

$$\Omega^\mu = y(1-y)\partial_y^2 + a(y)\partial_y + F^\mu(y), \quad (1.4)$$

such that

$$\Psi_d^\mu(y) \Omega^\mu = y(1-y)(\Psi_d^\mu)''(y) + a(y)(\Psi_d^\mu)'(y) + \Psi_d^\mu(y) F^\mu(y) = \Lambda_d^\mu \Psi_d^\mu(y), \quad (1.5)$$

where Λ_d^μ is a constant diagonal matrix, see for instance [15] or [22, Section 3]. Note that the expression (1.4) is available for any irreducible K -representation, see e.g. [25, Prop. 9.1.2.11]. The full spherical function Ψ_0^μ of degree zero is also a solution to the first-order differential equation,

$$y(1-y)\partial_y \Psi_0^\mu(y) = (S^\mu + yR^\mu)\Psi_0^\mu(y), \quad (1.6)$$

for certain constant matrices R^μ and S^μ , [22, Theorem 3.1]. Note that differentiating (1.6) yields a matrix-valued hypergeometric differential equation for Ψ_0^μ in the sense of [23]. Observe that (1.6) can be seen as a differential operator acting on the left of Ψ_0^μ while (1.5) is an operator acting on the right. One of the consequences of (1.6) is that Ψ_0^μ is invertible on $(0, 1)$, see [22, Cor. 3.4] so that

$$P_d^\mu(y) = \Psi_d^\mu(y) (\Psi_0^\mu(y))^{-1},$$

Now that the link between the spherical functions of type μ and the corresponding MVOPs is clear, we allow ourselves to drop the index μ from the notation. However, various constants and coefficients that occur later on, do depend on μ .

The matrix-valued polynomials P_d satisfy a three-term recurrence relation,

$$yP_d(y) = A_d P_{d+1}(y) + B_d P_d(y) + C_d P_{d-1}(y),$$

with A_d being an invertible diagonal matrix so that the leading coefficient of P_d is invertible for all $d \in \mathbb{N}_0$, see [15, Section 1]. Furthermore the orthogonality relations for the full spherical functions (1.3) imply that the polynomials P_d are orthogonal with respect to the pairing

$$\langle P, Q \rangle_W = \int_0^1 P(y) W(y) Q(y)^* dy,$$

where W is the weight matrix defined by

$$W(y) = (1-y)^\alpha y^\beta W_{pol}(y), \quad W_{pol}(y) = \Psi_0(y) T (\Psi_0(y))^*. \quad (1.7)$$

The first order differential equation (1.6) implies that the polynomials P_d are eigenfunctions of the hypergeometric differential operator

$$D = \Phi_0 \Omega (\Phi_0)^{-1} = y(y-1)\partial_y^2 + \partial_y(C - yU) - V, \quad (1.8)$$

where

$$C = \frac{\lambda_1 m}{rp^2(M-m)} - 2S, \quad U = 2R - \frac{\lambda_1}{rp^2}, \quad V = -\frac{\Lambda_0}{rp^2}.$$

Here M, m, p, r are constants related with the pair (G, K) and their values are given in [22, Table 2] for the various cases. The differential operator (1.8) is symmetric with respect to W so that the pair (W, D) is a MVCP. This fact follows from the symmetry of the Casimir operator Ω on G .

Deformation of MVOPs. We summarize the discussion above as follows: we have two pairs, a pair (w, Ω) together with a sequence of full spherical functions $(\Psi_d)_d$ and a matrix valued

classical pair (W, D) with a sequence of MVOPs $(P_d)_d$. These pairs are related by the function Ψ_0 , as follows:

$$W(y) = y^\alpha(1-y)^\beta \Psi_0(y) T(\Psi_0(y))^*, \quad D = \Psi_0 \Omega \Psi_0^{-1},$$

$$P_d = \Psi_d(\Psi_0)^{-1} \quad \forall d \in \mathbb{N}.$$

We shall refer to the functions, weights, differential operators on the spherical level and on the polynomial level, where the relation is given by the function Ψ_0 .

The Jacobi polynomials in a single variable can be given as a family of scalar valued orthogonal polynomials $(P_d^{(\alpha, \beta)})_d$, associated to the scalar valued classical pair $(w^{(\alpha, \beta)}, D^{(\alpha, \beta)})$. In this case, there is a two dimensional deformation that allows for the fundamental shifts

- $G_+(\alpha, \beta) = \partial_y$, the shift is $(1, 1)$,
- $G_-(\alpha, \beta) = 2y(y-1)\partial_y + (\alpha + \beta)(2y-1) + \alpha - \beta$, the shift is $(-1, -1)$,
- $E_+(\alpha, \beta) = y\partial_y + \beta$, the shift is $(1, -1)$,
- $E_-(\alpha, \beta) = (y-1)\partial_y + \alpha$, the shift is $(-1, 1)$.

This means, for example, that $(E_+(\alpha, \beta)P_{d+1}^{(\alpha, \beta)})_d$ is a family of scalar valued orthogonal polynomials associated to $(w^{(\alpha+1, \beta-1)}, D^{(\alpha+1, \beta-1)})$. The Jacobi polynomials with geometric parameters (i.e. (α, β) is associated to the root multiplicities of compact symmetric spaces) are examples of MVOPs, where $\Psi_0 = 1$. The theory of the shift operators is now an application of the Heckman-Opdam theory on the level of the spherical functions, and it translates in the above fashion to the shift operators of the Jacobi polynomials. This transition becomes more involved if we consider more general K -types, because Ψ_0 is then no longer trivial. This is why we content to study only deformations that allow for the shift operator ∂_y . We proceed in three steps.

- D1.** The deformed differential operators $D^{(\kappa)}$ have to satisfy $D^{(\kappa+1)} \circ \partial_y = \partial_y \circ D^{(\kappa)}$. In this way, whenever $(P_d^{(\kappa)})_d$ is a sequence of eigenfunctions of $D^{(\kappa)}$, then so is $(\partial_y P_d^{(\kappa)})_d$ for $D^{(\kappa+1)}$, with $\kappa \in \mathbb{N}_0$. The dependence on κ is polynomial, hence we may vary κ in \mathbb{C} by analytic continuation.
- D2.** The deformed differential operators $D^{(\kappa)}$ give rise differential operators on the level of spherical functions,

$$\Omega^{(\kappa)} := \Psi_0^{-1} \circ D^{(\kappa)} \circ \Psi_0 = y(1-y)\partial_y^2 + a^{(\kappa)}(y)\partial_y + F^{(\kappa)}(y).$$

We stipulate that $\Omega^{(\kappa)}$ be symmetric with respect to deformed weight

$$w^{(\kappa)}(y)dy = T^{(\kappa)}y^{\alpha(\kappa)}(1-y)^{\beta(\kappa)}dy, \quad (1.9)$$

with $T^{(\kappa)} > 0$ diagonal. This yields a equations for $T^{(\kappa)}, a^{(\kappa)}, F^{(\kappa)}, \alpha(\kappa)$ and $\beta(\kappa)$.

- D3.** The question remains, whether the pair $(W^{(\kappa)}, D^{(\kappa)})$ allows for the shift ∂_y . In other words, if $(P_d^{(\kappa)})_d$ is a family of MVOPs with respect to $W^{(\kappa)}$ we want to decide wether the sequence of derivatives $(\partial_y P_d^{(\kappa)})_d$ is orthogonal with respect to $W^{(\kappa+1)}$. Here we use criterion from Cantero, Moral and Velázquez [2], who characterized the sequences MVOPs whose derivatives yield a families of MVOPs.

Note that **D1** implies that $a^{(\kappa)}(y)$ is scalar valued function. We can now state our main result. Let (G, K, μ) be a triple as in [15], and let Ω be the radial part of the Casimir operator (1.4). Let T be the diagonal matrix introduced in (1.3). The associated matrix-valued classical pair (W, D) is given by (1.7) and (1.8).

Theorem 1.2. *For $\kappa \geq 0$ let $a^{(\kappa)}(y) = a(y) + \kappa(1-2y) = \beta + \kappa + 1 - y(\alpha + \beta + 2\kappa + 2)$, let $F^{(\kappa)}$ be given by*

$$F^{(\kappa)} = F - \kappa \Psi_0^{-1}(U + \kappa - 1)\Psi_0 - \kappa(1-2y)\Psi_0^{-1}\Psi'_0.$$

and let $T^{(\kappa)}$ be a solution of

$$T^{(\kappa)}(F^{(\kappa)}(y))^* = F^{(\kappa)}(y)T^{(\kappa)}. \quad (1.10)$$

Then the pair

$$W^{(\kappa)}(y) = (1-y)^{\alpha+\kappa}y^{\beta+\kappa}\Psi_0(y)T^{(\kappa)}\Psi_0(y)^*, \quad D^{(\kappa)} = \Psi_0^{-1}\Omega^{(\kappa)}\Psi_0, \quad (1.11)$$

is a MVCP, if $T^{(\kappa)} > 0$.

Remark 1.3. Using **D3** we show that several examples of MVCPs that we obtained in [22] admit a deformation that allows for the shift ∂_x . Moreover, the deformations of the MVCPs of [18, 19] that we obtain, are the same, up to a constant, as those in [17].

Remark 1.4. Note that $W^{(\kappa)} > 0$ almost everywhere if and only if $T(\kappa) > 0$. The conclusion that $W^{(\kappa)} > 0$ almost everywhere for the case studied in [17], depends on the LDU-decomposition of the weight. The method that we propose has the important advantage that its nature is simpler than the deformation considered in [17], because we go back to the spherical level, where the weight and the differential operator are in some sense much simpler.

In fact, the matrix $T = T^0$ corresponding to (G_0, K_0, μ_0) is a diagonal matrix whose entries are dimensions of the irreducible M_0 -modules that occur in the restriction of $\pi_{\mu_0}^{K_0}$ to M_0 (where $M_0 \subset K_0$ is a compact subgroup that we obtain after some choices, see [22]). In some of the examples, where the $T^{(k)}$ can be interpreted as the " $T^{(0)}$ " for other triples (G_k, K_k, μ_k) , our deformation is really an interpolation of all these dimensions.

The fact that we pass from the level of spherical functions to the level of polynomials, always using the same function Ψ_0 , is a restriction to the theory. However, this restriction is justified by the examples, where we also have families $(G_k, K_k, \mu_k)_{k \in \mathbb{N}}$ of Gelfand pairs with specified K_k -type μ_k , such that $\Psi_0^{\mu_k} = \Psi_0^{\mu_{k_0}}$, for some $k_0 \in \mathbb{N}$.

A second restriction to the theory is given by the shape of the weight matrices on the spherical level, that we insist to be of the form (1.9) with $T^{(\kappa)} > 0$ diagonal. This is justified by the same argument as before, namely that this is what happens in the examples.

Outlook. It would be interesting to investigate the effect of other shift operators on the MVOPs in a single variable. A possible approach is to study the group theoretic interpretation of the shift operators for MVOPs with geometric parameters. This may also give more insight to the result of Cantero, Moral and Velázquez [2] on the level of the spherical functions.

It would also be interesting to determine the generators for the algebra of differential operators that have the spherical functions as eigenfunctions. If the dependence of these generators on the root multiplicities is understood, one should investigate whether the whole algebra may be deformed, instead of just one of the differential operators.

Organization of the paper. In Section 2 we will work out **D1**, **D2** and derive equations on the level of the spherical functions that will in turn yield a deformation of the given MVCP. In view of **D3** we explain how it can be verified that the deformation allows for the shift ∂_y . For the cases where this holds true we derive a Rodrigues type formula.

In Section 3 we apply our construction to the family of matrix-valued Chebyshev polynomials obtained in [18, 19]. In Section 4 we apply our new method to the examples of 2×2 matrix-valued orthogonal polynomials obtained in [22]. We see that the families related to the groups $SU(n)$ and $SO(n)$ are maximal in the sense that they are closed under our deformations. On the other hand, the families associated to the symplectic groups $Sp(n)$ give rise to a new families with an extra free parameter.

2. DEFORMATION OF MVCPs

In this section we carry out **D1-D3** of the introduction. Let (w, Ω) and Ψ_0 be the input data coming from the representation theory of compact Gelfand pairs of rank one, and let (W, D) be the MVCP associated to this data.

Proposition 2.1. *Let Q be a polynomial on \mathbb{R} that satisfies*

$$y(1-y)Q''(y) + Q'(y)(C - yU) - Q(y)V = \Lambda Q(y),$$

where Λ is a constant matrix. Then the k -th derivative $\partial_y^k Q = Q^{(k)}$ satisfies

$$y(1-y)(Q^{(k)})''(y) + (Q^{(k)})'(y)(C + k - y(U + 2k)) - Q^{(k)}(y)(V + kU + k(k-1)) = \Lambda Q^{(k)}(y).$$

Proof. The verification, which is based on induction over k , is left to the reader. \square

The following Corollary follows immediately by analytic continuation and it settles **D1** of our program.

Corollary 2.2. *Given a differential operator $D = y(1-y)\partial_y^2 + \partial_y(C-yU) - V$, the differential operator $D^{(\kappa)} = y(1-y)\partial_y^2 + \partial_y(C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)}$ with $C^{(\kappa)} = C + \kappa$, $U^{(\kappa)} = U + 2\kappa$ and $V^{(\kappa)} = V + \kappa U + \kappa(\kappa-1)$, satisfies $\partial \circ D^{(\kappa)} = D^{(\kappa+1)} \circ \partial$ for all $\kappa \geq 0$.*

The deformation $(w^{(\kappa)}, \Omega^{(\kappa)})$ of (w, Ω) is of the form

$$\Omega^{(\kappa)} = y(1-y)\partial_y^2 + a^{(\kappa)}(y)\partial_y + F^{(\kappa)}(y), \quad w^{(\kappa)}(y) = y^{\beta(\kappa)}(1-y)^{\alpha(\kappa)}T^{(\kappa)}, \quad (2.1)$$

for certain functions $a^{(\kappa)}(y), F^{(\kappa)}(y)$, a constant matrix $T^{(\kappa)}$ and constants $\alpha(\kappa), \beta(\kappa)$ such that

$$a^{(0)}(y) = a(y), \quad F^{(0)}(y) = F(y), \quad T^{(0)} = T, \quad \alpha(0) = \alpha, \quad \beta(0) = \beta.$$

This leads to a deformed pair $(W^{(\kappa)}, D^{(\kappa)}) = (\Psi_0 w^{(\kappa)} \Psi_0^*, \Psi_0 \Omega^{(\kappa)} \Psi_0^{-1})$. To see how this deformation translates to the level of spherical functions, where it is easier to check the symmetry conditions, we use the following result.

Proposition 2.3. *Let $\Omega^{(\kappa)}$ be given by (2.1). The differential operator $D^{(\kappa)} = \Psi_0 \Omega^{(\kappa)} (\Psi_0)^{-1}$ is a matrix-valued hypergeometric operator of the form*

$$D^{(\kappa)} = y(1-y)\partial_y^2 + \partial_y(C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)},$$

for some constant matrices $C^{(\kappa)}, U^{(\kappa)}, V^{(\kappa)}$ if and only if $a^{(\kappa)}$ and $F^{(\kappa)}$ are given by

$$\begin{aligned} F^{(\kappa)}(y) &= -\Psi_0^{-1} V^{(\kappa)} \Psi_0 - y(1-y)\Psi_0^{-1} \Psi_0'' - a^{(\kappa)} \Psi_0^{-1} \Psi_0', \\ a^{(\kappa)}(y) &= (C^{(\kappa)} - yU^{(\kappa)}) - 2(S + yR), \end{aligned} \quad (2.2)$$

where S, R are the matrices given in (1.6).

Proof. By a straightforward computation, it is readily seen that $\Psi_0^{-1} \Omega^{(\kappa)} \Psi_0$ is the differential operator

$$y(1-y)\partial_y^2 + \partial_y[2y(1-y)\Psi_0' \Psi_0^{-1} + a^{(\kappa)}] + [y(1-y)\Psi_0'' \Psi_0^{-1} + a^{(\kappa)} \Psi_0' \Psi_0^{-1} + \Psi_0 F^{(\kappa)} \Psi_0^{-1}].$$

This is a matrix-valued hypergeometric operator if and only if there exist constant matrices $C^{(\kappa)}, U^{(\kappa)}$ and $V^{(\kappa)}$ such that

$$2y(1-y)\Psi_0' \Psi_0^{-1} + a^{(\kappa)} = C^{(\kappa)} - yU^{(\kappa)}, \quad y(1-y)\Psi_0'' \Psi_0^{-1} + a^{(\kappa)} \Psi_0' \Psi_0^{-1} + \Psi_0 F^{(\kappa)} \Psi_0^{-1} = V^{(\kappa)}. \quad (2.3)$$

In the first equation of (2.3) we use (1.6) to obtain

$$2(S + yR) + a^{(\kappa)}(y) = (C^{(\kappa)} - yU^{(\kappa)}).$$

Finally the second equation of (2.3) holds if and only if $F^{(\kappa)}$ is given by (2.2). \square

As an immediate consequence of our calculations, we obtain the following result, which in particular shows that $a^{(\kappa)}$ is a scalar function, whenever we deform D according to Corollary 2.2.

Corollary 2.4. *The deformed differential operator is of the form*

$$D^{(\kappa)} = y(1-y)\partial_y^2 + \partial_y(C^{(\kappa)} - yU^{(\kappa)}) - V^{(\kappa)},$$

with $C^{(\kappa)} = C + \kappa$, $U^{(\kappa)} = U + 2\kappa$ and $V^{(\kappa)} = V + \kappa U + \kappa(\kappa-1)$ if and only if

$$a^{(\kappa)}(y) = a(y) + \kappa(1-2y), \quad F^{(\kappa)}(y) = F(y) + \kappa \Psi_0^{-1}(U + \kappa - 1) \Psi_0 - \kappa(1-2y) \Psi_0^{-1} \Psi_0'.$$

We proceed to investigate the symmetry relations for the deformed differential operator on the level of the spherical functions.

Proposition 2.5. *Assume that the differential operator $D^{(\kappa)} = \Psi_0 \Omega^{(\kappa)} (\Psi_0)^{-1}$ is of the form of Corollary 2.4, then $D^{(\kappa)}$ is symmetric with respect to $W^{(\kappa)}$ if and only if*

$$(y(1-y)w^{(\kappa)}(y))' = a^{(\kappa)}(y)w^{(\kappa)}(y), \quad T^{(\kappa)}(F^{(\kappa)}(y))^* = F^{(\kappa)}(y)T^{(\kappa)}. \quad (2.4)$$

Proof. Let (P_d) be a sequence of MVOPs with respect to $W^{(\kappa)}$ and let $\Psi_d^{(\kappa)} = P_d \Psi_0$. The operator $D^{(\kappa)}$ is symmetric with respect to $W^{(\kappa)}$ if and only if

$$\begin{aligned} \int_0^1 (\Psi_d^{(\kappa)} \Omega^{(\kappa)})(y) T^{(\kappa)} (\Psi_{d'}^{(\kappa)})^* (1-y)^{\alpha(\kappa)} y^{\beta(\kappa)} dy &= \int_0^1 (P_d D^{(\kappa)})(y) W^{(\kappa)}(y) (P_{d'})^* dy \\ &= \int_0^1 P_d(y) W^{(\kappa)}(y) (P_{d'} D^{(\kappa)})^* dy \\ &= \int_0^1 \Psi_d^{(\kappa)}(y) T^{(\kappa)} (\Psi_{d'}^{(\kappa)} \Omega^{(\kappa)})^* (1-y)^{\alpha(\kappa)} y^{\beta(\kappa)} dy, \end{aligned}$$

where we used the symmetry condition (1.2) in the second and third equation. In other words, $D^{(\kappa)}$ is symmetric with respect to $W^{(\kappa)}$ if and only if

$$\int_0^1 (\Psi_d^{(\kappa)} \Omega^{(\kappa)})(y) T^{(\kappa)} (\Psi_{d'}^{(\kappa)})^* (1-y)^{\alpha(\kappa)} y^{\beta(\kappa)} dy = \int_0^1 \Psi_d^{(\kappa)}(y) T^{(\kappa)} (\Psi_{d'}^{(\kappa)} \Omega^{(\kappa)})^* (1-y)^{\alpha(\kappa)} y^{\beta(\kappa)} dy. \quad (2.5)$$

It follows from integration by parts that (2.5) holds true for all $d, d' \in \mathbb{N}$ if and only if

$$(y(1-y) w^{(\kappa)}(y))' = a^{(\kappa)}(y) w^{(\kappa)}(y), \quad T^{(\kappa)} (F^{(\kappa)}(y))^* = F^{(\kappa)}(y) T^{(\kappa)}. \quad (2.6)$$

This completes the proof of the proposition. \square

Remark 2.6. If $F^{(\kappa)}$ is a tridiagonal matrix and $F_{i,i+1}^{(\kappa)}/F_{i+1,i}^{(\kappa)}$ is constant for all $i = 0, \dots, N$, then the condition on the right of (2.4) is equivalent to

$$T_{i+1,i+1}^{(\kappa)} = \frac{F_{i+1,i}^{(\kappa)}}{F_{i,i+1}^{(\kappa)}} T_{i,i}^{(\kappa)},$$

so that $T^{(\kappa)}$ is determined up to a constant factor.

Note that (2.6) for $\kappa = 0$ implies $a(y) = 1 + \beta - (2 + \beta - \alpha)y$, so by (2.4) we have $a^{(\kappa)}(y) = 1 + \beta + \kappa - (2 + 2\kappa + \beta - \alpha)y$. Again by (2.4) we find $\alpha(\kappa) = \alpha + \kappa$ and $\beta(\kappa) = \beta + \kappa$.

We have now settled step **D2** of our program. The question remains, whether the deformation $(W^{(\kappa)}, D^{(\kappa)})$ allows for the shift ∂_y . In other words, we need to determine whether the sequence of derivatives $(\partial_y P_n^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa+1)}$. In [2] Cantero, Moral, Velázquez characterized the sequences of MVOPs whose derivatives are also orthogonal. If there exist matrix polynomials $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$ of degree two and one respectively such that

$$(W^{(\kappa)}(y) \Gamma_2^{(\kappa)}(y))' = W^{(\kappa)}(y) \Gamma_1^{(\kappa)}(y), \quad (2.7)$$

then the sequence of derivatives $(\partial_y P_n^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa)}(y) \Gamma_2^{(\kappa)}$. In our case, using the expression (1.9) for the weight matrices, we have that if

$$\Gamma_2^{(\kappa)}(y) = y(1-y) (\Psi_0(y)^*)^{-1} (T^{(\kappa)})^{-1} T^{(\kappa+1)} \Psi_0(y)^*, \quad (2.8)$$

$$\Gamma_1^{(\kappa)}(y) = (W^{(\kappa)}(y))^{-1} (W^{(\kappa+1)}(y))' \quad (2.9)$$

are polynomials of degree two and one respectively, then sequence $(\partial_y P_n^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa+1)} = W^{(\kappa)} \Gamma_2^{(\kappa)}$. We can rewrite (2.7) in terms of differential operators, as stated in the following proposition which summarizes this discussion.

Proposition 2.7. *The following are equivalent.*

- (1) *The deformed pair $(W^{(\kappa)}, D^{(\kappa)})$ allows for the shift ∂_y .*
- (2) *$\Gamma_2^{(\kappa)}$ is a matrix polynomial of degree two and $\Gamma_1^{(\kappa)}$ is a matrix polynomial of degree one.*
- (3) *The differential operator*

$$D_{(\Gamma_2, \Gamma_1)}^{(\kappa)} = \frac{d^2}{dy^2} \Gamma_2^{(\kappa)}(y)^* + \frac{d}{dy} \Gamma_1^{(\kappa)}(y)^*$$

is symmetric with respect to $W^{(\kappa)}$ and has the polynomials $P_d^{(\kappa)}$ as eigenfunctions.

Proof. 1 \iff 2 follows from [2].

2 \Rightarrow 3: If the sequence $(\partial_y P_d^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa+1)}$, then the equations (2.7) hold true for polynomials $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$ of degrees two and one respectively, see [2, Thm. 3.14]. A simple computation shows that (2.7) implies the conditions for the symmetry of [4, Thm. 3.1], we omit the proof.

3 \Rightarrow 1: Now assume that $D_{(\Gamma_2, \Gamma_1)}^{(\kappa)}$ is symmetric with respect to $W^{(\kappa)}$ and has the polynomials $P_n^{(\kappa)}$ as eigenfunctions. The proof is completely analogous to [17, Proposition 3.3]. Since $D_{(\Gamma_2, \Gamma_1)}^{(\kappa)}$ is symmetric, $\deg \Gamma_2^{(\kappa)} = 2$, $\deg \Gamma_1^{(\kappa)} = 1$ (see [11, Proposition 2.6]) and the conditions

$$(\Gamma_2^{(\kappa)}(y)^* W^{(\kappa)}(y))'' - (\Gamma_1^{(\kappa)}(y)^* W^{(\kappa)}(y))' = 0, \quad \lim_{y \rightarrow 0,1} \left(\Gamma_2^{(\kappa)}(y)^* W^{(\kappa)}(y) \right)' - \Gamma_1^{(\kappa)}(y)^* W^{(\kappa)}(y) = 0,$$

hold. If we integrate with respect to y we obtain $(\Gamma_2^{(\kappa)}(y)^* W^{(\kappa)}(y))' = \Gamma_1^{(\kappa)}(y)^* W^{(\kappa)}(y)$ which implies (2.7). \square

We have now settled all the steps **D1-D3** of our program. We lack a group theoretical interpretation for the matrix polynomials $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$.

Rodrigues formula. Now we assume that Proposition 2.7 holds true. One of the main consequences is the existence of a Rodrigues formula for the MVOPs. This fact was already noticed in [17, Theorem 3.1, (iii)] and the proof in our case follows the same lines. We take the Hilbert C^* -module $\mathcal{H}^{(\kappa)}$ given by the completion of the space of matrix-valued orthogonal polynomials with respect to the matrix-valued inner product $\langle \cdot, \cdot \rangle^{(\kappa)}$ defined by $W^{(\kappa)}$. The lowering operator $\partial_y : \mathcal{H}^{(\kappa)} \rightarrow \mathcal{H}^{(\kappa+1)}$ is an unbounded operator with dense domain and dense range. This operator has an adjoint, which is a raising operator preserving polynomials, that can be calculated explicitly in terms of $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$.

Lemma 2.8. *Let $\kappa > 0$ and define the first order differential operator $\Xi^{(\kappa)}$ given by*

$$Q\Xi^{(\kappa)} = \partial_y Q(\Gamma_2^{(\kappa)})^* + Q(\Gamma_1^{(\kappa)})^*.$$

Then $\langle \partial_y P, Q \rangle^{(\kappa+1)} = -\langle P, Q\Xi^{(\kappa)} \rangle^{(\kappa)}$ for matrix-valued polynomials P and Q .

Proof. This is analogous to [17, Corollary 2.5]. \square

Theorem 2.9. *Let $\kappa > 0$ and let $(Q_n^{(\kappa)})_n$ be the sequence of monic orthogonal polynomials with respect to $W^{(\kappa)}$. Then there exists a constant matrices $G_n^{(\kappa)}$ such that*

$$Q_n(y) = G_n^{(\kappa)} (\partial_y^n W^{(\kappa+n)})(W^{(\kappa)}(y))^{-1}. \quad (2.10)$$

for all $n \in \mathbb{N}$.

Proof. Is analogous to [17, Theorem 3.1, (iii)]. \square

Corollary 2.10. *The following integral formula holds,*

$$Q_n(y) = \frac{G_n^{(\kappa)}}{2\pi i} \left[\int_{\gamma(x)} \frac{W^{(\kappa+n)}(z)}{(z-y)^n} dz \right] (W^{(\kappa)}(y))^{-1},$$

where γ is a closed contour around y .

Proof. The proof follows by applying Cauchy's integral formula to (2.10). \square

Corollary 2.11. *The following relation holds for the monic orthogonal polynomials Q_n*

$$G_n^{(\kappa)} (G_{n+1}^{(\kappa)})^{-1} Q_{n+1}^{(\kappa)} = (\partial_y Q_n^{(\kappa+1)})(\Gamma_2^{(\kappa)})^* + Q_n^{(\kappa+1)}(\Gamma_1^{(\kappa)})^*. \quad (2.11)$$

Proof. It follows from (2.10) that

$$Q_n^{(\kappa+1)}(y) W^{(\kappa+1)}(y) = G_n^{(\kappa)} (\partial_y^n W^{(\kappa+n+1)}(y)).$$

If we take the derivative with respect to y on both sides of this equation we obtain

$$\partial_y(Q_n^{(\kappa+1)}W^{(\kappa+1)}) = G_n^{(\kappa)}(\partial_y^{n+1}W^{(\kappa+n+1)}(y)) = G_n^{(\kappa)}(G_{n+1}^{(\kappa)})^{-1}Q_{n+1}^{(\kappa)}W^{(\kappa)}. \quad (2.12)$$

On the other hand we have

$$\partial_y(Q_n^{(\kappa+1)}W^{(\kappa+1)}) = (\partial_y Q_n^{(\kappa+1)})W^{(\kappa+1)} + Q_n^{(\kappa+1)}(\partial_y W^{(\kappa+1)}). \quad (2.13)$$

By combining (2.12) and (2.13) and multiplying by $(W^{(\kappa)})^{-1}$ on the right, we obtain

$$G_n^{(\kappa)}(G_{n+1}^{(\kappa)})^{-1}Q_{n+1}^{(\kappa)} = (\partial_y Q_n^{(\kappa+1)})W^{(\kappa+1)}(W^{(\kappa)})^{-1} + Q_n^{(\kappa+1)}(\partial_y W^{(\kappa+1)})(W^{(\kappa)})^{-1}.$$

Using (2.8) and (2.9) gives the result. \square

Corollary 2.12. *The matrices $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$ can be written in terms of the monic polynomials $P_n^{(\kappa)}$ and the coefficients $G_n^{(\kappa)}$ in the following way*

$$\begin{aligned} (\Gamma_1^{(\kappa)})^* &= G_0^{(\kappa)}(G_1^{(\kappa)})^{-1}Q_1^{(\kappa)}, \\ (\Gamma_2^{(\kappa)})^* &= G_1^{(\kappa)}(G_2^{(\kappa)})^{-1}Q_2^{(\kappa)} - G_0^{(\kappa+1)}(G_1^{(\kappa+1)})^{-1}(\Gamma_1^{(\kappa+1)})^*(\Gamma_1^{(\kappa)})^*. \end{aligned}$$

Proof. The corollary follows by evaluating (2.11) in $n = 0$ and $n = 1$. \square

3. MATRIX-VALUED GEGENBAUER POLYNOMIALS

The goal of this section is to deform the family of matrix-valued Chebyshev polynomials obtained in [18, 19] from the study of spherical functions associated to the pair $(G, K) = (\text{SU}(2) \times \text{SU}(2), \text{diag SU}(2))$. We will show that our construction is an alternative to the one given in [17] and provides a different factorization for the weight matrix. The key factorization in [17] is the LDU decomposition of the weight matrix, which plays a fundamental role, for instance to show that the weight matrix is positive definite, see [17, Corollary 2.5]. In our case, this is a direct consequence of the decomposition (1.11) of the deformed weight, as we noticed in Remark 1.4.

In this case, we have all the ingredients to perform the deformation explicitly. For $\ell \in \frac{1}{2}\mathbb{N}$, the full spherical functions $\Phi_d : [0, 4\pi] \rightarrow \mathbb{C}^{(2\ell+1) \times (2\ell+1)}$ were introduced in [18, Definition 2.2, Theorem 2.1 and (2.5)]. In particular, the full spherical function of degree zero is given by

$$(\Phi_0(t))_{n,m} = \sum_{j_1=-\frac{n}{2}}^{\frac{n}{2}} \sum_{j_2=-\frac{2\ell-n}{2}}^{\frac{2\ell-n}{2}} \delta_{-\ell+m, j_1+j_2} \binom{n}{j_1+\frac{n}{2}} \binom{2\ell-n}{j_2+\frac{(2\ell-n)}{2}} \binom{2\ell}{2\ell-m}^{-1} e^{i(j_2-j_1)t}. \quad (3.1)$$

If we denote $\Psi_d(y) = \Phi_d(\arccos(1-2y))$, the full spherical polynomial of degree d is given by

$$P_d(y) = \Psi_d(y)(\Psi_0(y))^{-1}, \quad y \in [0, 1].$$

For $\ell = 0$, it boils down to a 1×1 matrix and it is a multiple of the Chebyshev polynomial of degree d . The polynomials $(P_d)_d$ form a sequence of matrix-valued orthogonal polynomials with non-singular leading coefficients, see [18, Proposition 4.6], with respect to the sesqui-linear pairing

$$\langle P, Q \rangle = \int_0^1 P(y)W(y)Q(y)^* dy, \quad W(y) = y^{1/2}(1-y)^{1/2}\Psi_0(y)\Psi_0(y)^*.$$

The full spherical functions Ψ_d satisfy the following differential equation,

$$\Psi_d(y)\Omega = y(1-y)\Psi_d''(y) + a(y)\Psi_d'(y) + \Psi_d(y)F(y) = \Lambda_d\Psi_d(y),$$

where $a(y) = \frac{3}{2} - 3y$, the eigenvalue is the diagonal matrix $(\Lambda_d)_{i,i} = -d(2\ell+2+d)I + i(2\ell-i)$, and

$$\begin{aligned} F(y) &= \sum_{i=0}^{2\ell} \frac{2y(1-y)(\ell(\ell+2) - i^2 + 2\ell i) - \ell(2i+1) + i^2}{2y(1-y)} E_{i,i} \\ &\quad + \sum_{i=1}^{2\ell} \frac{i(2\ell-i+1)(1-2y)}{4y(1-y)} E_{i,i-1} + \sum_{i=0}^{2\ell-1} \frac{(i+1)(2\ell-i)(1-2y)}{4y(1-y)} E_{i,i+1}. \end{aligned} \quad (3.2)$$

The differential operator Ω is the radial part of the Casimir operator on G , see [19, Section 7.2]. The function Ψ_0 satisfies the first order differential equation

$$2y(1-y)\Psi'_0(y) + (S - \ell + 2\ell y)\Psi_0(y) = 0, \quad (3.3)$$

where S is the tridiagonal matrix

$$S = \sum_{i=1}^{2\ell} \frac{i}{2} E_{i,i-1} + \sum_{i=0}^{2\ell-1} \frac{(2\ell-i)}{2} E_{i,i+1}, \quad (3.4)$$

see [19, Lemma 7.12]. The polynomials P_d are joint eigenfunctions of the matrix-valued differential operators D and E given by

$$D = y(1-y)\frac{d^2}{dy^2} + \left(\frac{d}{dy}\right)(C - yU) - V, \quad E = \left(\frac{d}{dy}\right)(yB_1 + B_0) + A_0,$$

where the matrices C , U , V , B_0 , B_1 and A_0 are given by

$$\begin{aligned} C &= -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{2} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(2\ell+3)}{2} E_{ii} - \sum_{i=0}^{2\ell} \frac{i}{2} E_{i,i-1}, \quad U = (2\ell+3)I, \\ V &= -\sum_{i=0}^{2\ell} i(2\ell-i)E_{i,i} \quad A_0 = \sum_{i=0}^{2\ell} \frac{(2\ell+2)(i-2\ell)}{2\ell} E_{i,i}, \quad B_1 = -\sum_{i=0}^{2\ell} \frac{(\ell-i)}{\ell} E_{i,i}, \\ B_0 &= -\sum_{i=0}^{2\ell} \frac{(2\ell-i)}{4\ell} E_{i,i+1} + \sum_{i=0}^{2\ell} \frac{(\ell-i)}{2\ell} E_{ii} + \sum_{i=0}^{2\ell} \frac{i}{4\ell} E_{i,i-1}. \end{aligned}$$

The first order differential equation (3.3) can be used to derive a simple and compact expression for Ψ_0 which will be crucial in the forthcoming subsections. The proof is relegated to the Appendix. Let K be the constant matrix

$$K_{i,j} = K_j(i) = K_j(i, 1/2, 2\ell), \quad i, j \in \{0, \dots, 2\ell\}. \quad (3.5)$$

where $K_n(x, p, N)$ are the Krawtchouk polynomials, see e.g. [16, §1.10]. The orthogonality relations for the Krawtchouk polynomials give a simple inverse for the matrix K , namely

$$K^{-1} = 2^{-2\ell} M K M,$$

where M is the diagonal matrix with entries $M_{j,j} = \binom{2\ell}{j}$, $j = 0, \dots, 2\ell$.

Theorem 3.1. *For any $\ell \in \frac{1}{2}\mathbb{N}$, we have*

$$\Psi_0(y) = K \Upsilon(y) K, \quad (3.6)$$

where K is the constant matrix given by (3.5) and Υ is the diagonal matrix

$$\Upsilon(y)_{j,j} = (-1)^{\frac{3j}{2}} \binom{2\ell}{j} y^{\frac{j}{2}} (1-y)^{\frac{2\ell-j}{2}}.$$

3.1. The deformation. In this subsection we apply Theorem 1.2 to obtain a deformation of the pair (W, D) . Since we want to deform matrix-valued Chebyshev polynomials into matrix-valued Gegenbauer polynomials, we shift $\kappa = \nu - 1$ in order to match the standard convention for Gegenbauer polynomials. In this way, our deformed polynomials $P_d^{(\nu)}$ coincide with the polynomials P_d for $\nu = 1$. We take

$$a^{(\nu)}(y) = \frac{1}{2} + \nu - y(2\nu + 1), \quad F^{(\nu)} = F - (\nu - 1)(2\ell + \nu + 1) - (\nu - 1)(1 - 2y)\Psi_0^{-1}\Psi'_0,$$

where F is given in (3.2). We follow the method described in Section 2. It follows from the explicit expression in Theorem 3.1 that

$$\Psi_0^{-1}\Psi'_0 = -\frac{1}{2y(1-y)}(S^* - \ell + 2\ell y).$$

Therefore we have

$$F^{(\nu)}(y) = F(y) - (\nu - 1)(2\ell + \nu + 1) + \frac{(\nu - 1)(1 - 2y)}{2y(1 - y)}(S^* - \ell + 2\ell y). \quad (3.7)$$

Note that $F^{(\nu)}$ is a tridiagonal matrix. Therefore we can use Remark 2.6 to obtain a diagonal matrix $T^{(\nu)}$ as in Theorem 1.2.

Lemma 3.2. *Let $T^{(\nu)}$ be the diagonal matrix*

$$T_{i,i}^{(\nu)} = \binom{2\ell}{i} \frac{(\nu)_i}{(\nu + 2\ell - i)_i}, \quad (3.8)$$

for $i = 0, \dots, \lfloor \ell \rfloor$ and $T_{i,i} = T_{2\ell-i, 2\ell-i}$. Then

$$T^{(\nu)} (F^{(\nu)}(y))^* = F^{(\nu)}(y) T^{(\nu)}.$$

Proof. It follows from (3.7) and (3.2) that

$$F_{i,i-1}^{(\nu)} = (2\ell - i + 1)(\nu - i + 1) \frac{(1 - 2y)}{y(1 - y)}, \quad F_{i,i-1}^{(\nu)} = (i + 1)(2\ell + \nu - i - 1) \frac{(1 - 2y)}{y(1 - y)}.$$

By Remark 2.6, if we define $T^{(\nu)}$ as the diagonal matrix

$$T_{i+1,i+1}^{(\nu)} = \frac{F_{i+1,i}^{(\nu)}}{F_{i,i+1}^{(\nu)}} T_{i,i}^{(\nu)} = \frac{(2\ell - i + 1)(\nu - i - 1)}{i(2\ell + \nu - i)} T_{i,i}^{(\nu)},$$

then the condition in Theorem 1.2 holds true. This completes the proof of the lemma. \square

Corollary 3.3. *If $T^{(\nu)}$ is as in Lemma 3.8, then the pair*

$$W^{(\nu)}(y) = (1 - y)^{\nu-1/2} y^{\nu-1/2} \Psi_0(y) T^{(\nu)} \Psi_0(y)^*,$$

$$D^{(\nu)} = y(1 - y)\partial_y^2 + \partial_y(C + \nu - 1 - y(U + 2\nu + 1)) - (V + (\nu - 1)U + (\nu - 1)(\nu - 2)),$$

is a MVCP.

Remark 3.4. Observe that the differential operator $2D^{(\nu)}$ is, up to a change of variables $x = 1 - 2y$ and a multiple of the identity, the differential operator $D^{(\nu)}$ in [17, Theorem 2.3]. In the following subsection we will show that the weight matrix $W^{(\kappa)}$ is also closely related to the weight matrix introduced in [17, Definition 2.1].

Remark 3.5. In [24] the authors construct families of matrix-valued orthogonal polynomials of size 2×2 and 3×3 from the study of spherical functions of fundamental K -types associated with the pair $(G, K) = (\text{SO}(n + 1), \text{SO}(n))$. If we restrict our deformed weight $W^{(\kappa)}$ to the size 3×3 , it can be matched with the one given in [24, Section 9.2] by the identification $\kappa = 1/(l + 1)$.

3.2. The shift operator ∂_y . The goal of this subsection is to prove that the pair $(W^{(\kappa)}, D^{(\kappa)})$ allows for the shift ∂_y . For this we will show that $\Gamma_2^{(\nu)}$ and $\Gamma_1^{(\nu)}$ defined as in (2.8) and (2.9) are matrix-valued polynomials of degree two and one respectively. Then by Proposition 2.7, if $(Q_n^{(\kappa)})_n$ is the sequence of monic orthogonal polynomials with respect to $W^{(\kappa)}$, we have that $\partial_y Q_d^{(\nu)} = dQ_d^{(\nu+1)}$, since the sequence of monic orthogonal polynomials is unique. The proof will follow from the explicit expression of Ψ_0 given in Theorem 3.1 and involves manipulation of Krawtchouk polynomials. Some of the formulas for Krawtchouk polynomials that are necessary in the proof are collected in the Appendix.

Proposition 3.6. *The functions $\Gamma_2^{(\nu)}$ and $\Gamma_1^{(\nu)}$ introduced in (2.8) and (2.9) are matrix polynomials of degree two and one respectively.*

Proof. First we compute the constant matrix $\Delta = 2^{-2\ell} M K M (T^{(\nu)})^{-1} T^{(\nu+1)} K$. Note that the i -th diagonal element of $(T^{(\nu)})^{-1} T^{(\nu+1)}$ is $(\nu + 2\ell - i)(\nu + i)/(\nu(\nu + 2\ell))$, so that

$$\Delta_{k,j} = \frac{2^{-2\ell}}{\nu(\nu + 2\ell)} \sum_{i=0}^{2\ell} \binom{2\ell}{i} \binom{2\ell}{k} (\nu + 2\ell - i)(\nu + i) K_i(k) K_j(i).$$

It follows from the relations (A.8), (A.10) and (A.11), that

$$\begin{aligned} \nu(\nu + 2\ell) \Delta = & - \sum_{k=0} \frac{(k+1)(k+2)}{4} E_{k,k+2} + \sum_{k=0} (\ell(\ell - 1/2) + k(k/2 - \ell) + \nu(\nu + 2\ell)) E_{k,k} \\ & - \sum_{k=0} \frac{(2\ell - k + 1)(2\ell - k + 2)}{4} E_{k,k-2}. \end{aligned} \quad (3.9)$$

First we show that $\Gamma_2^{(\nu)}$ is a polynomial of degree two. It follows from Proposition 3.1 that

$$\Gamma_2^{(\nu)}(y) = y(1 - y) K^{-1} \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y) K.$$

It follows directly from the explicit expressions of Δ and Υ that $y(1 - y)\bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y)$ is a polynomial of degree two and, therefore, so is $\Gamma_2^{(\nu)}$.

Now we prove that $\Gamma_1^{(\nu)}$ is a polynomial of degree one. Note that

$$\begin{aligned} K \Gamma_1^{(\nu)}(y) K^{-1} = & - \frac{(2y - 1)(2\nu + 1)}{2} \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y) - 2^{-2\ell-1} \bar{\Upsilon}(y)^{-1} M K M N K \bar{\Upsilon}(y) \\ & - \frac{\ell(2y - 1)}{2} \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y) + y(1 - y) \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y)', \end{aligned} \quad (3.10)$$

where $N = (T^{(\nu)})^{-1} S^* T^{(\nu+1)}$. A simple computation shows that N is the tridiagonal matrix given by

$$N_{i,i-1} = \frac{i(\nu + 2\ell - i)(\nu + 2\ell - i + 1)}{2\nu(\nu + 2\ell)}, \quad N_{i,i} = 0, \quad N_{i,i+1} = \frac{(2\ell - i)(\nu + i)(\nu + i + 1)}{2\nu(\nu + 2\ell)}.$$

Using the explicit expressions of N and K we obtain:

$$\begin{aligned} 2^{-2\ell} (M K M N K)_{k,j} = & \frac{\binom{2\ell}{k}}{2\nu(\nu + 2\ell)} \sum_{i=0}^{2\ell} \binom{2\ell}{i} [(2\ell - i)(\nu + i)(\nu + i + 1) K_k(i) K_j(i + 1) \\ & + i(2\ell + \nu - i)(2\ell + \nu - i + 1) K_k(i) K_j(i - 1)] \end{aligned}$$

In the formula above, we replace the terms $(2\ell - i)K_j(i + 1)$ and $iK_j(i - 1)$ using Lemma A.4 and we get an expression that is be evaluated using (A.10) and (A.11). We obtain

$$\begin{aligned} 2^{-2\ell-1} (M K M N K)_{k,j} = & \frac{(2\nu + 3\ell - k + 1)(k + 1)(k + 2)}{8\nu(\nu + 2\ell)} \delta_{j,k+2} \\ & + \frac{(k - \ell)(k^2 - 2\ell k - l + 1 - 4\nu\ell - 2\nu^2 - 2\ell^2)}{4\nu(\nu + 2\ell)} \delta_{j,k} - \frac{(2\ell - k + 1)(2\ell - k + 2)(\ell + 2\nu + k - 1)}{8\nu(\nu + 2\ell)} \delta_{j,k-2}. \end{aligned}$$

It follows from (3.10) and the equation above that $(K \Gamma_1^{(\nu)}(y) K^{-1})_{i,j} = 0$ unless $j = k - 2, k, k + 2$. Moreover, a straightforward computation using the explicit expressions of Δ , Υ and $2^{-2\ell} M K M N K$ shows that $(K \Gamma_1^{(\nu)}(y) K^{-1})_{i,j}$ is a polynomial of degree one. This completes the proof of the proposition. \square

In order to relate the deformed family $W^{(\nu)}$ with the family introduced in [17] we need the following corollary.

Corollary 3.7. *The polynomial $\Gamma_2^{(\nu)}$ is given explicitly by*

$$\begin{aligned} \frac{4\kappa(\kappa+2\ell)}{\ell^2} \Gamma_2^{(\nu)}(x) &= (1-2y)^2 \sum_{i=0}^{2\ell} \frac{(\ell-i)^2}{\ell^2} E_{i,i} - 4y(1-y) \frac{(\ell+\nu)^2}{\ell^2} \\ &+ (1-2y) \sum_{i=1}^{2\ell} \frac{(i-1-2\ell)(2\ell-2i+1)}{2\ell^2} E_{i,i-1} + (1-2y) \sum_{i=0}^{2\ell-1} \frac{(i+1)(2\ell-2i-1)}{2\ell^2} E_{i,i+1} \\ &+ \sum_{i=2}^{2\ell} \frac{(2\ell-i+2)(2\ell-i+1)}{4\ell^2} E_{i,i-2} + \sum_{i=0}^{2\ell} \frac{-i(2\ell-i+1) - (2\ell-i)(i+1)}{4\ell^2} E_{i,i} \\ &+ \sum_{i=0}^{2\ell-2} \frac{(i+2)(i+1)}{4\ell^2} E_{i,i+2}. \end{aligned}$$

Remark 3.8. Observe that, up to the change of variables $x = 1-2y$ and a constant $4\kappa(\kappa+2\ell)/\ell^2$, the matrix Γ_2 coincides with the polynomial of degree two Φ given in [17, (4.9)].

Proof. The corollary is proven by a straightforward computation. From the proof of Proposition 3.6, we have that

$$K \Gamma_2^{(\nu)}(y) = y(1-y) \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y) K. \quad (3.11)$$

Now the proof follows by a tedious but direct verification of (3.11). Using the explicit expression of Δ given in (3.9) the right hand side of (3.11) becomes

$$\begin{aligned} y(1-y) \bar{\Upsilon}(y)^{-1} \Delta \bar{\Upsilon}(y) &= \sum_{j=2}^{2\ell} y^2 \frac{(2\ell-j-1)(2\ell-j)}{4\nu(\nu+2\ell)} E_{j,j+2} \\ &+ \sum_{j=0}^{2\ell} y(1-y) \frac{(\ell(\ell-1/2) + j(j/2-\ell) + \nu(\nu+2\ell))}{\nu(\nu+2\ell)} E_{j,j} + \sum_{j=0}^{2\ell-2} (1-y)^2 \frac{j(j-1)}{4\nu(\nu+2\ell)} E_{j,j-2}. \end{aligned}$$

Therefore, both sides of (3.11) are polynomials of degree two. The proof of the corollary follows by showing that the coefficients of degree 0,1,2 on the left and right hand sides of (3.11) coincide. We include a sketch of the proof for the coefficient of degree two and the other cases are analogous. The coefficient of y^2 of the (j, k) -th entry of (3.11) is given by

$$\begin{aligned} -(\nu+j)(\nu+2\ell-j)K_j(k) &= \frac{j(j-1)}{4} K_k(j-2) - (\ell(\ell-1/2) + j(j/2-\ell) + \nu(\nu+2\ell))K_k(j) \\ &+ \frac{(2\ell-j-1)(2\ell-j)}{4} K_k(j+2). \end{aligned}$$

Now we apply (A.12) and (A.13) twice on the first and last terms of the right hand side, and the three term recurrence relation on the left hand side and the middle term of the right hand side. The equation above becomes an expression involving Krawtchouk polynomials $K_h(j)$ for $h = k-2, k-1, k, k+1, k+2$ with coefficients which are independent of j . The equation is then verified by checking the coefficients of the Krawtchouk polynomials of different degrees. The remaining cases follow in an similar way. \square

Recall that our deformed family of weight matrices $W^{(\nu)}$ coincides with the one given in [17, Definition 2.1] for $\nu = 1$. Since $W^{(\nu+1)} = W^{(\nu)}\Gamma_2^{(\nu)}$, see (2.8), in view of [17, Theorem 2.4], it follows that the weight matrix in [17, Definition 2.1] coincides up to a constant with $W^{(\nu)}$ for all integer values of ν . From a continuation argument we conclude that the two families of weights coincide up to constant multiple.

Remark 3.9. Note that the operator $E^{(\nu)} = E + \nu(A_0 + B_1)$ satisfies $\partial \circ E^{(\nu)} = E^{(\nu+1)} \circ \partial$. Since $[E^{(\nu+1)}, D^{(\nu+1)}] \circ \partial = \partial \circ [E^{(\nu)}, D^{(\nu)}]$, and $[E, D] = 0$ see [18, 19], it follows that $[E^{(\nu)}, D^{(\nu)}] = 0$

for all $\nu \in \mathbb{N}$. Moreover for any $\nu \geq 0$ and any smooth $\mathbb{C}^{2\ell+1}$ -valued function F we have

$$\begin{aligned} F(D^{(\nu)}E^{(\nu)} - E^{(\nu)}D^{(\nu)}) &= \nu(1-2y)(\partial_y F)E + \nu(FE)(U + \nu - 1) \\ &\quad + \nu(FD)(A_0 + B_1) - \nu(F(A_0 + B_1))D - \nu(1-2y)\partial_y \nu(FE) - \nu(FE)(U + \nu - 1). \end{aligned} \quad (3.12)$$

Now (3.12) is a polynomial function in ν which is zero for infinitely many values of ν and thus it is zero for all ν . Therefore we have

$$[E^{(\nu)}, D^{(\nu)}] = 0, \quad \text{for all } \nu > 0.$$

Since $E^{(\nu)}$ commutes with $D^{(\nu)}$, it follows that the monic orthogonal polynomials $P_n^{(\nu)}$ are eigenfunctions of $E^{(\nu)}$, i.e.

$$P_n^{(\nu)}E^{(\nu)} = \Lambda_n(E^{(\nu)})P_n^{(\nu)}, \quad \text{for all } n \in \mathbb{N},$$

where $\Lambda_n(E^{(\nu)}) = nB_1 + A_0 + \nu(B_1 + A_0)$. Observe that since the eigenvalues $\Lambda_n(E^{(\nu)})$ are diagonal with real entries, and thus Hermitian, the differential operator $E^{(\nu)}$ is symmetric with respect to $W^{(\nu+1)}$, see [11, Corollary 4.5]. Observe that the differential operator $E^{(\nu)}$ coincides with the differential operators in [17, Corollary 4.1].

4. EXAMPLES OF DIMENSION TWO

In this section we apply our method to two of the examples given in [22, §1]. These examples are related to the Gelfand pairs $(\mathrm{SU}(n+1), \mathrm{U}(n))$ and $(\mathrm{USp}(2n), \mathrm{USp}(2n-2) \times \mathrm{USp}(2))$ and correspond to matrix-valued orthogonal polynomials of Jacobi type.

4.1. Case a1. Let us consider the Gelfand pair $(G, K) = (\mathrm{SU}(n+1), \mathrm{U}(n))$. This example is studied in [22, Page 7, case a1]. For $n \geq 2$ we have $\alpha = n-1$, $\beta = 0$ and two free parameters $1 \leq i \leq n-1$ and $m \in \mathbb{N}$. We have the following expressions,

$$\Psi_0^{(n,m,i)}(y) = y^{\frac{m}{2}} \begin{pmatrix} y^{\frac{1}{2}} & 1 \\ y^{\frac{1}{2}} & \frac{(m+1)-y(m+n-i+1)}{i-n} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n-i}{i} \end{pmatrix}, \quad (4.1)$$

where we indicate the free parameters as superscripts. Moreover, we have $a(y) = 1 - y(n+1)$ and

$$F(y) = \begin{pmatrix} (1+m)(1+m+2n) + \frac{(i-n)}{(1-y)} - \frac{(1+m)^2}{4y} & \frac{i\sqrt{y}}{1-y} \\ \frac{(i-n)\sqrt{y}}{1-y} & \frac{m(y(m+2n)-m)}{4y} - \frac{4iy}{1-y} \end{pmatrix}$$

and the differential operator $D = y(1-y)\partial_y^2 + \partial_y(C - yU) - V$ is determined by

$$\begin{aligned} C &= \begin{pmatrix} \frac{(m+1)(-m-n-2+i)}{-n-m-1+i} & \frac{-n+i}{-n-m-1+i} \\ -\frac{m+1}{-n-m-1+i} & \frac{-m^2+2i-2n+mi-2m-mn-1}{-n-m-1+i} \end{pmatrix}, \quad U = \begin{pmatrix} n+m+2 & 0 \\ -1 & n+m+3 \end{pmatrix}, \\ V &= \begin{pmatrix} 0 & 0 \\ 0 & n+m+1-i \end{pmatrix}. \end{aligned}$$

As in Theorem 1.2, we define $F^{(\kappa)} = F + \kappa\Psi_0^{-1}(U + \kappa - 1)\Psi_0 - \kappa(1-2y)\Psi_0^{-1}\Psi_0'$. By a straightforward computation we verify that

$$F^{(\kappa)}(y)_{0,1} = -\frac{\sqrt{y}(\kappa+i)}{y-1}, \quad F^{(\kappa)}(y)_{1,0} = \frac{\sqrt{y}(-n+i)}{y-1},$$

so that by Remark 2.6, the matrix $T^{(\kappa)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n-i}{i+\kappa} \end{pmatrix}$, is a solution to (1.10). Therefore Theorem 1.2 implies that $(W^{(n,m,i,\kappa)}, D^{(n,m,i,\kappa)})$ with

$$W^{(n,m,i,\kappa)}(y) = y^\kappa(1-y)^{n-1+\kappa}\Psi_0^{n,m,i}(y)T^{(\kappa)}(\Psi_0^{n,m,i}(y))^* \quad (4.2)$$

$$D^{(n,m,i,\kappa)} = y(1-y)\partial_y^2 + \partial_y(C + \kappa - y(U + 2\kappa)) - (V + \kappa U + \kappa(\kappa-1)),$$

is a MVCP. It is a straightforward computation that the functions

$$\Gamma_2^{(\kappa)} = (\Psi_0(y)^*)^{-1}(T^{(\kappa)})^{-1}T^{(\kappa)}\Psi_0(y)^*, \quad \Gamma_1^{(\kappa+1)} = (W^{(n,m,i,\kappa)}(y))^{-1}(W^{(n,m,i,\kappa+1)}(y))',$$

are matrix-valued polynomials of degrees two and one respectively. We omit this computation. It then follows that Proposition 2.7 and Theorem 2.9 apply to this case. Therefore the monic orthogonal polynomials $(Q_d^{(\kappa)})_d$ with respect to $W^{(\kappa)}$ satisfy

$$\partial_y Q_d^{(n,m,i,\kappa)}(y) = d Q_{d-1}^{(n,m,i,\kappa+1)}(y),$$

and the following Rodrigues formula holds

$$Q_d(y) = G_d (\partial_y^d W^{(n,m,i,\kappa+d)}) (W^{(n,m,i,\kappa)})^{-1},$$

for certain constant matrix G_d .

Remark 4.1. For nonnegative integer values of κ , it follows directly from the form of $T^{(\kappa)}$ and (4.2) that $W^{(n,m,i,\kappa)} = W^{(n+\kappa,m+\kappa,i+\kappa)}$ so that

$$\partial_y Q_d^{(n,m,i)}(y) = d Q_{d-1}^{(n+1,m+1,i+1)}(y).$$

Remark 4.2. The case $m \in \mathbb{Z}_{<0}$ is given in [22, Page 7, case **a2**]. The matrix $\Psi_0 = \Psi_0^{m<0}$ is given by

$$\Psi_0^{m<0}(y) = \begin{pmatrix} \frac{m-y(m-i)}{i} & y^{\frac{1}{2}} \\ 1 & y^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_0^{(n,m-1,i+n)}(y) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\Psi_0^{(n,-m-1,n-i)}$ is given in (4.1). The case **a2** is therefore contained into the case **a1**.

4.2. Case c1. Now we consider the Gelfand pair $(G, K) = (\mathrm{Sp}(2n), \mathrm{Sp}(2n-2) \times \mathrm{Sp}(2))$. This example is studied in [22, Page 7, case **a1**]. For $n \geq 3$ we have $\alpha = 2n-3$, $\beta = 1$. We have the following expressions

$$\Psi_0(y) = \begin{pmatrix} \sqrt{y} & 1 \\ \sqrt{y} & \frac{y(n-1)-1}{n-2} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & n-2 \end{pmatrix}.$$

We also have $a(y) = 2 - 2yn$,

$$F(t) = \begin{pmatrix} \frac{4y^2n+4yn-y^2-18y+3}{4y(y-1)} & -\frac{2\sqrt{y}}{y-1} \\ -\frac{2\sqrt{y}(n-2)}{y-1} & \frac{2y}{y-1} \end{pmatrix},$$

and the matrix-valued differential operator $D = y(1-y)\partial_y^2 + \partial_y, (C - yU) - V$, where

$$C = \begin{pmatrix} \frac{2n-1}{n-1} & \frac{n-2}{n-1} \\ \frac{1}{n-1} & \frac{3n-4}{n-1} \end{pmatrix}, \quad U = \begin{pmatrix} 2n+1 & 0 \\ -1 & 2n+2 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 0 & 2n-2 \end{pmatrix}.$$

If we define $F^{(\kappa)}$ as in Theorem 1.2, a straightforward verification shows that the unique solution $T^{(\kappa)}$ to (1.10) normalized by $T^{(0)} = T$ is given by

$$T^{(\kappa)} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2(n-2)}{\kappa+2} \end{pmatrix}.$$

Now Theorem 1.2 says that $(W^{(\kappa)}, D^{(\kappa)})$ with

$$\begin{aligned} W^{(\kappa)}(y) &= y^{\alpha+\kappa} (1-y)^{\beta+\kappa} \Psi_0(y) T^{(\kappa)} \Psi_0(y)^*, \\ &= y^{(\kappa+1)} (1-y)^{(\kappa+2n-3)} \begin{pmatrix} 2 \frac{2y+y\kappa+2n-4}{2+\kappa} & 2 \frac{2yn-2+y\kappa}{2+\kappa} \\ 2 \frac{2yn-2+y\kappa}{2+\kappa} & 2 \frac{y^2n^2-4y^2n-2yn+\kappa yn+2y^2-2y\kappa+2}{(2+\kappa)(n-2)} \end{pmatrix}, \\ D^{(\kappa)} &= y(1-y)\partial_y^2 + \partial_y (C + \kappa - y(U + 2\kappa)) - (V + \kappa U + \kappa(\kappa-1)), \end{aligned}$$

is a MVCP. Moreover, if we compute explicitly the functions $\Gamma_2^{(\kappa)}$ and $\Gamma_1^{(\kappa)}$ given in (2.8) and (2.8) respectively, we obtain

$$\begin{aligned}\Gamma_2^{(\kappa)} &= y^2 \begin{pmatrix} -1 & -(3+\kappa)^{-1} \\ 0 & -\frac{2+\kappa}{3+\kappa} \end{pmatrix} + y \begin{pmatrix} \frac{\kappa n-1+2n-\kappa}{(3+\kappa)(n-1)} & \frac{1}{(3+\kappa)(n-1)} \\ \frac{n-2}{(3+\kappa)(n-1)} & \frac{\kappa n+3n-4-\kappa}{(3+\kappa)(n-1)} \end{pmatrix}, \\ \Gamma_1^{(\kappa)} &= y \begin{pmatrix} -2n-1-2\kappa & -\frac{1+\kappa}{3+\kappa} \\ 0 & -2\frac{(2+\kappa)(\kappa+n+1)}{3+\kappa} \end{pmatrix} \\ &\quad + \begin{pmatrix} \frac{-3-\kappa^2+6n-2\kappa+4\kappa n+\kappa^2 n}{(3+\kappa)(n-1)} & \frac{3+2\kappa}{(3+\kappa)(n-1)} \\ \frac{(n-2)(2n+1+2\kappa)}{(3+\kappa)(n-1)} & \frac{\kappa^2 n+7n+6\kappa n-8\kappa-10-\kappa^2}{(3+\kappa)(n-1)} \end{pmatrix}.\end{aligned}$$

so that $\Gamma_2^{(\kappa)}$ is a polynomial of degree two and $\Gamma_1^{(\kappa)}$ is a polynomial of degree one. Therefore, it follows from Proposition 2.7 that the sequence of derivatives $(\partial_y P_d^{(\kappa)})$ is orthogonal with respect to $W^{(\kappa+1)}$.

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APPENDIX A.

Here we give a proof of the explicit expression for the function Ψ_0 in terms of Krawtchouk polynomials. We will use that Ψ_0 is a solution to the equation (3.3) and the fact that the matrix S can be diagonalized explicitly. Note that the columns of K are precisely the different eigenvectors of the matrix S from (3.4). More precisely we have

$$S \cdot K = \text{diag}(-\ell, -\ell+1, \dots, \ell-1, \ell) \cdot K,$$

see [19, Lemma 4.3] for the proof. Observe that the columns of Ψ_0 are solutions to the first order differential equation

$$y(1-y) F'(y) + \frac{1}{2} (S - \ell + 2\ell y) F(y) = 0, \quad (\text{A.1})$$

Proposition A.1. For $j = 0, 1, \dots, 2\ell$, let v_j be the eigenvector of S with eigenvalue $\ell - j$. Then

$$F_j(y) = y^{\frac{j}{2}} (1-y)^{\frac{2\ell-j}{2}} v_j,$$

is a solution of (A.1). Moreover, $\{F_j\}_{j=0}^{2\ell}$ is a basis of solutions of (A.1).

Proof. This follows by replacing F_j in (A.1). The set $\{F_j\}_{j=0}^{2\ell}$ is basis of solutions since v_j are all linearly independent. \square

A simple calculation shows that

$$F_j((1 - \cos(t))/2) = 2^{-2\ell} (-1)^{\frac{j}{2}} e^{-i\ell t} (e^{it} - 1)^j (e^{it} + 1)^{2\ell-j} v_j. \quad (\text{A.2})$$

Remark A.2. It follows directly from the previous proposition that each entry of $F_j(\cos t)$ is a linear combination of $\{e^{-i\ell t}, e^{-i(\ell-1)t}, \dots, e^{i\ell t}\}$. Moreover, for all $j = 0, 1, \dots, 2\ell$, we have

$$F_j((1 - \cos(t))/2) = \left(2^{-2\ell} (-1)^{\frac{3j}{2}} e^{-i\ell t} + \sum_{k=1}^{2\ell} a_k e^{i(-\ell+k)t} \right) v_j, \quad (\text{A.3})$$

for certain coefficients a_k .

Lemma A.3. Let Φ_0 be given by (3.1). Then

$$\Phi_0 = M^{-1} e^{-i\ell t} + \sum_{k=1}^{2\ell} A_k e^{i(-\ell+k)t}, \quad M = \sum_{j=0}^{2\ell} \binom{2\ell}{j} E_{j,j},$$

where A_k , $k = 1, \dots, 2\ell$ are $(2\ell+1) \times (2\ell+1)$ matrices.

Proof. The proof follows directly from (3.1). \square

Proof of Theorem 3.1. Since Ψ_0 is a solution of (A.1) and $\{F_j\}$ is a basis of solutions, every column of Ψ_0 is a linear combination of the functions F_j , $j = 0, \dots, 2\ell$. If we denote by $\Psi_0^{(k)}$ the k -th column of Ψ_0 , then there exist constants $a_0^{(k)}, \dots, a_{2\ell}^{(k)}$ such that

$$\Psi_0^{(k)}(y) = a_0^{(k)} F_0(y) + \dots + a_{2\ell}^{(k)} F_{2\ell}(y).$$

If we make the change of variables variable $\cos t = 1 - 2y$, using that $\Psi_0^{(k)}(\arccos(1 - 2y)) = \Phi_0^{(k)}(t)$, it follows from (A.2) that

$$\Phi_0^{(k)}(t) = 2^{-2\ell} (-1)^{\frac{j}{2}} e^{-i\ell t} [a_0^{(k)} (e^{it} + 1)^{2\ell} v_0 + \dots + (-1)^\ell a_{2\ell}^{(k)} (e^{it} - 1)^{2\ell} v_{2\ell}] \quad (\text{A.4})$$

If we look at the coefficient of $e^{-i\ell t}$ on both sides of (A.4), using (A.3) and Lemma A.3, we obtain the following equation,

$$\binom{2\ell}{k}^{-1} e_k = 2^{-2\ell} \sum_{j=0}^{2\ell} a_j^{(k)} (-1)^{\frac{3j}{2}} v_j, \quad (\text{A.5})$$

where e_k is the standard basis vector. Let Γ be the diagonal matrix given by $\Gamma_{j,j} = (-1)^{\frac{3j}{2}}$. Then (A.5) can be written in matrix form as

$$\binom{2\ell}{k}^{-1} e_k = 2^{-2\ell} K \Gamma (a_0^{(k)}, \dots, a_{2\ell}^{(k)})^t.$$

Then it follows that

$$(a_0^{(k)}, \dots, a_{2\ell}^{(k)})^t = 2^\ell \binom{2\ell}{2\ell - k}^{-1} \Gamma^{-1} K^{-1} e_k.$$

In matrix form, (A.4) can be written as

$$\tilde{\Phi}_0^{(k)}(t) = K \tilde{\Upsilon}(t) M (a_0^{(k)}, \dots, a_{2\ell}^{(k)})^t, \quad (\text{A.6})$$

where $\tilde{\Upsilon}$ is the diagonal matrix

$$\tilde{\Upsilon}(t)_{j,j} = 2^{-\ell} (-1)^{\frac{j}{2}} \binom{2\ell}{i} e^{-i\ell t} (e^{it} - 1)^j (e^{it} + 1)^{2\ell-j}.$$

If we replace this expression in (A.6) we obtain

$$\tilde{\Phi}_0^{(k)}(t) = 2^{2\ell} \binom{2\ell}{2\ell - k}^{-1} K \tilde{\Upsilon}(t) \Gamma^{-1} K^{-1} e_k,$$

which leads to

$$\Psi_0^{(k)}(y) = 2^{2\ell} K \Upsilon(y) M K^{-1} M^{-1} e_k, \quad (\text{A.7})$$

The column vector (A.7) is precisely the k -th column of (3.6). This completes the proof of the theorem. \square

We proceed to collect results and formulas about Krawtchouk polynomials that we use in Section 3. We will need the following relations

$$\sum_{i=0}^{2\ell} \binom{2\ell}{i} K_n(i) K_m(i) = \delta_{n,m} 2^{2\ell} \binom{2\ell}{n}^{-1}, \quad (\text{A.8})$$

$$-i K_n(i) = \frac{N-n}{2} K_{n+1}(i) - \frac{N}{2} K_n(i) + \frac{n}{2} K_{n-1}(i). \quad (\text{A.9})$$

Using the three-term recurrence relation (A.9) and orthogonality (A.8), we obtain

$$\sum_{i=0}^{2\ell} \binom{2\ell}{i} i K_k(i) K_j(i) = 2^{2\ell} \binom{2\ell}{k}^{-1} \begin{cases} -\frac{k+1}{2}, & j = k+1, \\ \ell, & j = k, \\ -\frac{2\ell-k+1}{2}, & j = k-1, \\ 0, & \text{otherwise,} \end{cases} \quad (\text{A.10})$$

and

$$\sum_{i=0}^{2\ell} \binom{2\ell}{i} i^2 K_k(i) K_j(i) = 2^{2\ell} \binom{2\ell}{k}^{-1} \begin{cases} \frac{(k+1)(k+2)}{4}, & j = k+2, \\ -\ell(k+1), & j = k+1, \\ (\ell(\ell+1/2) + k(\ell-k/2)), & j = k, \\ -\ell(2\ell-k+1), & j = k-1, \\ \frac{(2\ell-k+1)(2\ell-k+2)}{4}, & j = k-2, \\ 0, & \text{otherwise} \end{cases} \quad (\text{A.11})$$

Lemma A.4. *The following recurrence relations hold:*

$$(2\ell - i)K_k(i+1) = \frac{1}{2}(2\ell - k)K_{k+1}(i) + (\ell - k)K_k(i) - \frac{1}{2}kK_{k-1}(i), \quad (\text{A.12})$$

$$iK_k(i-1) = -\frac{1}{2}(2\ell - k)K_{k+1}(i) + (\ell - k)K_k(i) + \frac{1}{2}kK_{k-1}(i) \quad (\text{A.13})$$

Proof. Note that we only have to prove one of the relations, because the relations add up to the difference equation [16, (1.10.5)]. Alternatively, replace i by $2\ell - i$ in the first equation and apply the basic relation ${}_2F_1\left(\begin{smallmatrix} a, b \\ c \end{smallmatrix}; z\right) = (1-z)^{-a} {}_2F_1\left(\begin{smallmatrix} a, c-b \\ c \end{smallmatrix}; z/(z-1)\right)$ specialized in $z = 2$. The basic relations

$$(2\ell - i) {}_2F_1\left(\begin{smallmatrix} -i-1, -k \\ -2\ell \end{smallmatrix}; z\right) + (i+k-2\ell) {}_2F_1\left(\begin{smallmatrix} -i, -k \\ -2\ell \end{smallmatrix}; z\right) = -k(z-1) {}_2F_1\left(\begin{smallmatrix} -i, -k+1 \\ -2\ell \end{smallmatrix}; z\right),$$

$$(2\ell - k) {}_2F_1\left(\begin{smallmatrix} -i, -k-1 \\ -2\ell \end{smallmatrix}; z\right) + (2k-2\ell + (i-k)z) {}_2F_1\left(\begin{smallmatrix} -i, -k \\ -2\ell \end{smallmatrix}; z\right) = -k(z-1) {}_2F_1\left(\begin{smallmatrix} -i, -k+1 \\ -2\ell \end{smallmatrix}; z\right),$$

imply the result. Indeed, subtracting one half times the second from the first yields

$$(2\ell - i) {}_2F_1\left(\begin{smallmatrix} -i-1, -k \\ -2\ell \end{smallmatrix}; z\right) = \frac{1}{2}(2\ell - k) {}_2F_1\left(\begin{smallmatrix} -i, -k-1 \\ -2\ell \end{smallmatrix}; z\right) + (\ell - \frac{1}{2}zk + (\frac{1}{2}z-1)i) {}_2F_1\left(\begin{smallmatrix} -i, -k \\ -2\ell \end{smallmatrix}; z\right) - \frac{1}{2} {}_2F_1\left(\begin{smallmatrix} -i, -k+1 \\ -2\ell \end{smallmatrix}; z\right),$$

which specializes to the desired equation for $z = 2$. \square

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